ECE 346: Digital Signal Processing — Spring 2021
Soo Min Kwon — Week of February 1, 2021

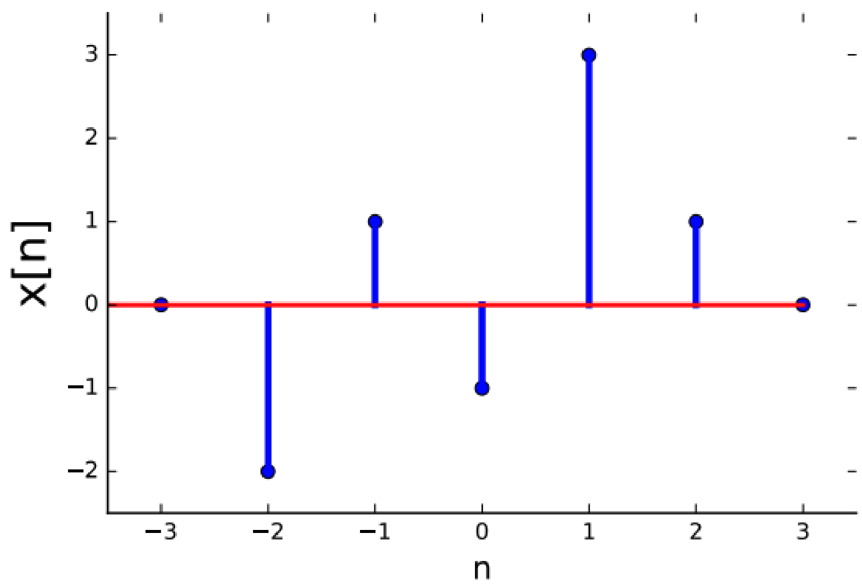
Recitation #2: Review of LSS Concepts

Objective & Outline

The objective of this week's recitation session is to review LSS concepts such as linearity, time invariance, convolution, and the CTFT/DTFT. The following is the outline of this solution guide:

1. Problems $1-4$: recitation problems
2. Problem 5: self-assessment problem

Problem 1 (Dealing with DT Signals)**.** Consider the following DT signal $x[n]$:



(a) Write $x[n]$ in terms of delta functions.

(b) Suppose a DT system has impulse response given by

$$h[n] = 2\delta[n] - \delta[n-1] + \delta[n-1] \tag{1}$$

Find the output of this system, y[n], given that the input is $x[n]$.

Solution:

(a) This part should be fairly straightforward:

$$x[n] = -2\delta[n+2] + \delta[n+1] - \delta[n] + 3\delta[n-1] + \delta[n-2] \tag{2}$$

(b) Here, we are asked to compute the convolution of $x[n]$ and $h[n]$ (i.e. $y[n] = x[n] * h[n]$):

$$\begin{aligned} y[n] &= 2[-2\delta[n+2] + \delta[n+1] - \delta[n] + 3\delta[n-1] + \delta[n-2]] && (3)\\ &-[-2\delta[n+1] + \delta[n] - \delta[n-1] + 3\delta[n-2] + \delta[n-3]] && (4)\\ &[-2\delta[n] + \delta[n-1] - \delta[n-2] + 3\delta[n-3] + \delta[n-4]] && (5)\\ &= -4\delta[n+2] + 4\delta[n+1] - 5\delta[n] + 8\delta[n-1] - 2\delta[n-2] + 2\delta[n-3] + \delta[n-4]. && (6) \end{aligned}$$

Note that all we're doing here is scaling $x[n]$ and shifting it accordingly to $h[n]$.

Problem 2 (Computing the CTFT)**.** Consider the following LTI system with impulse response

$$h(t) = u(t). \tag{7}$$

(8)

Using the CTFT, do the following:

(a) Find the output of the system with the input $x_1(t) = \cos(t)$.

(b) Find the output of the system with the input $x_2(t) = \cos(2t)$.

Solution:

We can first compute $H(j\Omega)$ using CTFT formula:

$$H(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega} \tag{9}$$

Now, we can use the fact that complex exponentials are eigenfunctions of LTI systems and compute the output based on their corresponding frequencies.

(a) We can "euler-ize" the cosine into complex exponentials:

$$\cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} \tag{10}$$

Evaluating the system at $\Omega = \pm 1$ yields

$$H(j) = \pi\delta(1) + \frac{1}{j} = -j, \tag{11}$$

$$H(-j) = \pi\delta(-1) + \frac{1}{-j} = j, \tag{12}$$

$$y_1(t) = \frac{-j}{2}e^{jt} + \frac{j}{2}e^{-jt} = \sin(t) \tag{13}$$

(b) This is essentially the same problem, except we are evaluating at $\Omega = \pm 2$:

$$H(2) = \pi\delta(2) + \frac{1}{j2} = \frac{-j}{2}, \tag{14}$$

$$H(-2) = \pi\delta(-2) + \frac{1}{-j2} = \frac{j}{2}, \tag{15}$$

$$y_2(t) = \frac{-j}{4}e^{j2t} + \frac{j}{4}e^{-j2t} = \frac{1}{2}\sin(2t) \tag{16}$$

Problem 3 (Filtering and the CTFT)**.** Consider the following CT signal

$$x(t) = 1 + 2\cos(5\pi t) + 3\sin(8\pi t) \tag{17}$$

that is an input to an LTI system with impulse response given by

$$h(t) = \frac{\sin(10\pi t)}{\pi t} - \frac{\sin(3\pi t)}{\pi t}. \tag{18}$$

Determine the output $y(t)$ of this system and plot the magnitude and phase of $Y(j\Omega)$.

Solution:

We can first "euler-ize" and then take the CTFT of $x(t)$:

$$X(j\Omega) = \delta(\Omega) + 2\pi\delta(\Omega + 5\pi) + 2\pi\delta(\Omega - 5\pi) + j3\pi\delta(\Omega + 8\pi) - j3\pi\delta(\Omega - 8\pi) \tag{19}$$

For $H(j\Omega)$, you should realize that this is actually the difference of two ideal low pass filters (LPFs) resulting in a bandpass filter. Thus,

$$H(j\Omega) = \begin{cases} 1, & 3\pi \leq |\Omega| \leq 10\pi \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

Now, if we compute $Y(j\Omega) = X(j\Omega)H(j\Omega)$, we would filter out the $\delta(\Omega)$ term and get

$$X(j\Omega) = 2\pi\delta(\Omega + 5\pi) + 2\pi\delta(\Omega - 5\pi) + j3\pi\delta(\Omega + 8\pi) - j3\pi\delta(\Omega - 8\pi), \tag{21}$$

with the following $y(t)$:

$$y(t) = 2\cos(5\pi t) + 3\sin(8\pi t). \tag{22}$$

The plots are shown at the end of this solution guide.

Problem 4 (Computing the DTFT)**.** Compute the DTFT of the following signals:

(a) $x_1[n] = u[n-2] - u[n-6]$

(b) $x_2[n] = n$ for $|n| \leq 3$ and 0 otherwise

(c) $x_3[n] = \sin(\frac{\pi}{2}n) + \cos(n)$

Solution:

(a) One approach is to apply the definition and factor out $e^{-j2\omega}$:

$$
\begin{aligned}
X_1(e^{j\omega}) &= e^{-j2\omega}\sum_{k=0}^{3} e^{-j\omega k} && (23)\\
&= e^{-j2\omega}\frac{1-e^{-j4\omega}}{1-e^{j\omega}} && (24)\\
&= e^{-j3.5\omega}\frac{e^{j2\omega}-e^{-j2\omega}}{e^{j0.5\omega}-e^{-j0.5\omega}} && (25)\\
&= e^{-j3.5\omega}\frac{\sin(2\omega)}{\sin(\omega/2)}. && (26)
\end{aligned}
$$

(b) By simply plugging into the DTFT formula:

$$
\begin{aligned}
X_2(e^{j\omega}) &= -3e^{j3\omega} - 2e^{j2\omega} - 1e^{j\omega} + 1e^{-j\omega} + 2e^{-2j\omega} + 3e^{-j3\omega} && (27)\\
&= -6j\sin(3\omega) - 4j\sin(2\omega) - 2j\sin(\omega). && (28)
\end{aligned}
$$

(c) This one we can use the table:

$$
X_3(e^{j\omega}) = -\frac{\pi}{j}\delta(\omega+\pi/2) + \frac{\pi}{j}\delta(\omega+\pi/2) + \pi\delta(\omega+1) + \pi\delta(\omega-1). \tag{29}
$$

Problem 5 (Self-assessment)**.** Try to solve each problem by yourself first, and then discuss with your group.

1. **DT Signals.** Determine the fundamental period and fundamental angular frequency of the following signal:

$$
x[n] = 3\cos(0.56\pi n + 1) \tag{30}
$$

2. **DT Convolution.** Consider the following two sequences $x_1[n]$ and $x_2[n]$:

$$
\begin{aligned}
x_1[n] &= u[n] - u[n-3] && (31)\\
x_2[n] &= \sum_{i=0}^{2}\delta[n-2i] && (32)
\end{aligned}
$$

Compute the following:

(a) Compute the convolution $y_1[n] = x_1[n] * x_2[n]$.

(b) Compute the convolution $y_2[n] = x_1[n-2] * x_2[n]$.

3. **Inverse DTFT.** Compute the inverse DTFT of the following signals:

(a) $X(e^{j\omega}) = 1$ for $\pi/4 \le |\omega| \le 3\pi/4$ and 0 otherwise

(b) $X(e^{j\omega}) = \cos^2(\omega) + \sin^2(3\omega)$

Solution:

1. For a DT sinusoid of the form $A\cos(\Omega n + \theta)$, the fundamental period is

$$N_0 = \frac{2\pi k}{\Omega}, \tag{33}$$

where k is the smallest integer value that results in an integer value for N_0. If such k does not exist, then it is not periodic. Further, the fundamental angular frequency is

$$\Omega_0 = \frac{2\pi}{N_0} \tag{34}$$

Using these formulas:

$$N_0 = \frac{25k}{7}, \tag{35}$$

and selecting $k = 7$ would yield $N_0 = 25$ samples. Further,

$$\Omega_0 = \frac{2\pi}{25} \tag{36}$$

rad/sample.

2. Note that

$$x_1[n] = \{1, 1, 1\} \tag{37}$$
$$x_2[n] = \{1, 0, 1, 0, 1\}. \tag{38}$$

Using DT convolution, this would yield the following:

$$y_1[n] = \{1, 1, 2, 1, 2, 1, 1\} \tag{39}$$
$$y_2[n] = \{0, 0, 1, 1, 2, 1, 2, 1, 1\} \tag{40}$$

The trick for the second problem (b) was to realize that we can just shift the answer of part (a) twice to the right (i.e. $y_2[n] = y_1[n-2]$).

3. For this problem:

   (a) This is an ideal bandpass filter (BPF) (draw a picture). We can solve this a couple of different ways, but let's try direct integration first:

$$x[n] = \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega = \frac{1}{\pi n}\left(\sin\left(\frac{3\pi}{4}n\right) - \sin\left(\frac{\pi}{4}n\right)\right). \tag{41}$$

   Another way to think about this is that it's an ideal LPF with cutoff $3\pi/4$ minus an ideal LPF with cutoff $\pi/4$:

$$x[n] = \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n} - \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{42}$$

Yet another way to do it would be to think of it as an 2 times an ideal LPF with cutoff $\pi/4$ modulated by a cosine at $\pi/2$.

The key to solving this one is to first sketch the spectrum (draw a picture) and then figure out how to build that picture from simpler things you know, like an ideal LPF.

(b) This one requires some trigonometry identities, or you can just use Euler:

$$X(e^{j\omega}) = \left(\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}\right)^2 + \left(\frac{1}{2j}e^{j3\omega} - \frac{1}{2j}e^{-j3\omega}\right)^2 \tag{43}$$

$$= \frac{1}{4}e^{-j2\omega} + \frac{1}{2} + \frac{1}{4}e^{j2\omega} - \frac{1}{4}e^{-j6\omega} - \frac{1}{4}e^{j6\omega} + \frac{1}{2} \tag{44}$$

$$= 1 - \frac{1}{4}e^{-j6\omega} + \frac{1}{4}e^{-j2\omega} + \frac{1}{4}e^{j2\omega} - \frac{1}{4}e^{j6\omega} \tag{45}$$

So now we can use the previous trick and read these off as a sum of delta functions:

$$x[n] = -\frac{1}{4}\delta[n-6] + \frac{1}{4}\delta[n-2] + \delta[n] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n+6]. \tag{46}$$

Plot to Problem 3: